On the sums of series of reciprocals*

Leonhard Euler

§1 The series of the reciprocals of the powers of natural numbers have already been treated and investigated so often that it seems almost impossible to discover anything new about them. For, almost everyone, who meditated about the sums of series, also tried to find the sums of series of this kind and has nevertheless not been able to express them by any method in an appropriate way. Even I, after I had given various summation methods[1], persecuted these series diligently and nevertheless have achieved nothing more than to define their true sums either approximately or reduce them to quadratures of highly transcendental curves; I did the first in my dissertation red last[2], but the latter in the preceding ones[3]. But here I will consider only the series of fractions, whose numerators are 1, but the denominators on the other hand are either squares or cubes or other powers of the natural numbers; one series of this kind is

$$1+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+\frac{1}{25}+\text{etc.},$$

likewise

*Original title: „De summis serierum reciprocarum“, first published in „*Commentarii academiae scientiarum Petropolitanae* 7, 1740, pp. 123-134“, reprinted in „*Opera Omnia*: Series 1, Volume 14, pp. 73 - 86“, Eneström-Number E41, translated by: Alexander Aycock for the project „Euler-Kreis Mainz“

[1] Euler refers to his papers “De summatione innumerabilium progressionum“ and “Methodus generalis summandi progressiones“, which are E20 and E25 in the Eneström-Index respectively.

[2] This refers to E25.

[3] He does so in E20.

$$1+\frac{1}{8}+\frac{1}{27}+\frac{1}{64}+\text{etc.}$$

and similar ones of the higher powers, whose general terms are expressed by this form $\frac{1}{x^n}$.

§2 But I was recently unexpectedly led to an elegant expression for the sum of this series

$$1+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+\text{etc.},$$

which depends on the quadrature of the circle so that, if one would have the true sum of this series, hence at the same time the quadrature of the circle[4] would follow. For, I found that six times the sum of this series is equal to the square of the circumference of the circle, whose diameter is 1, or having put the sum of this series $= s$, the ratio of $\sqrt{6s}$ to 1 will be the the same as the ratio of circumference to the diameter of a circle. But I showed recently that the sum of this series approximately is

$$1.6449340668482264364;$$

if this number is multiplied by 6 and then the square root is taken, indeed the following number will result

$$3.141592653589793238$$

expressing the circumference of this circle whose diameter is 1. Further, following the same path, on which I was led to this sum, I also detected the sum of this series

$$1+\frac{1}{16}+\frac{1}{81}+\frac{1}{256}+\frac{1}{625}+\text{etc.}$$

to depend on the quadrature of the circle. For, its sum multiplied by 90 gives the forth power of the circumference of the circle, whose diameter is 1. And in like manner I was also able to determine the sums of the following series, in which the exponents of the powers are even numbers.

[4]By this Euler does not mean the geometrical construction of π but only an explicit for it.

§3 Therefore, to show in the most convenient way, how I obtained all these summations, I want to explain everything I did and applied in order. In the circle (Fig. 1) $AMBNA$ described around the center C with the radius BC or $BC = 1$ I contemplated an arbitrary arc AM, whose sine is MP, but its cosine is CP. Now having put the arc $AM = s$, the sine $PM = y$ and the cosine $CP = x$ one is able to define so the sine y as the cosine x from the given arc s in terms of a power series using a well-known method[5] ; for, it is, as one can see in various sources,

$$y = s - \frac{s^3}{1 \cdot 2 \cdot 3} + \frac{s^5}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} - \frac{s^7}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7} + \text{etc.}$$

and

$$x = 1 - \frac{s^2}{1 \cdot 2} + \frac{s^4}{1 \cdot 2 \cdot 3 \cdot 4} - \frac{s^6}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6} + \text{etc.}$$

Considering these equations I got to the sum of the series of reciprocals mentioned above; both of these equations lead almost to the same results, and therefore it will be sufficient to have treated only the one in the way, which I will explain here.

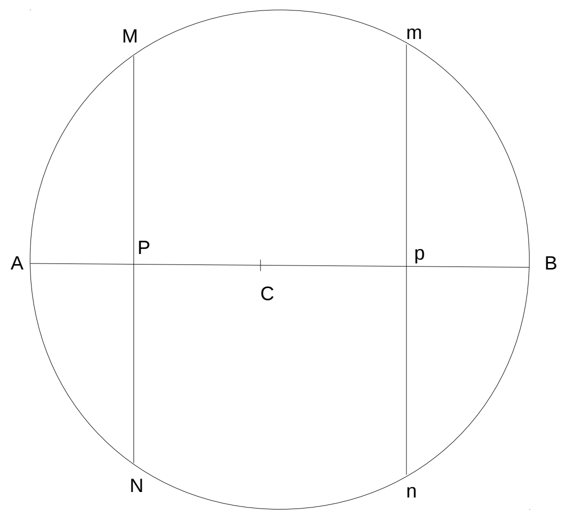

Fig. 1

[5]By this Euler means the Taylor series expansion.

§4 Therefore, the first equation

$$y = s - \frac{s^3}{1 \cdot 2 \cdot 3} + \frac{s^5}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} - \frac{s^7}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7} + \text{etc.}$$

expresses the relation among the arc and the sine. Hence using this relation it will be possible to determine the arc from a given sine and vice versa the corresponding sine from a given arc. But in this dissertation I consider the sine y as given and investigate, how the arc s must be found from y. Here, it is especially to be considered that innumerable arcs correspond to the same sine y; the propounded equation will hence have to yield these innumerable arcs. If in this equation s is considered as an unknown quantity, it has infinitely many dimensions and hence it is not surprising, if this equation contains innumerable simple factors; and each single one set equal to zero must give an appropriate value for s.

§5 But as, if all factors of this equation would be known, also all its roots or all values of s would be known, so vice versa, if all values of s can be assigned, one will then have all factors of the series. But in order to be able to see so the roots as the factors more clearly, I transform the propounded equation into this form

$$0 = 1 - \frac{s}{y} + \frac{s^3}{1 \cdot 2 \cdot 3y} - \frac{s^5}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5y} + \text{etc.}$$

If now all roots of this equation or all arcs, whose sine is the same, namely y, were A, B, C, D, E etc., then the factors will be these quantities

$$1 - \frac{s}{A}, \quad 1 - \frac{s}{B}, \quad 1 - \frac{s}{C}, \quad 1 - \frac{s}{D} \quad \text{etc.}$$

Therefore, it will be

$$1 - \frac{s}{y} + \frac{s^3}{1 \cdot 2 \cdot 3y^3} - \frac{s^5}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5y} + \text{etc.} = \left(1 - \frac{s}{A}\right)\left(1 - \frac{s}{B}\right)\left(1 - \frac{s}{C}\right)\left(1 - \frac{s}{D}\right) \text{etc.}$$

§6 But from the nature and the resolution of equations it is known that the coefficient of the term, in which s is contained, or $\frac{1}{y}$ is equal to the sum of all coefficients of s in the factors or

$$\frac{1}{y}=\frac{1}{A}+\frac{1}{B}+\frac{1}{C}+\frac{1}{D}+\text{etc.}$$

Further, the coefficient of s^2, which is $=0$ because of the missing term in the series, is equal to the aggregate of products of two terms of the series $\frac{1}{A}$, $\frac{1}{B}$, $\frac{1}{C}$, $\frac{1}{D}$ etc. Further, $-\frac{1}{1\cdot 2\cdot 3y}$ will be equal to the aggregate of products of three factors of the same series $\frac{1}{A}$, $\frac{1}{B}$, $\frac{1}{C}$, $\frac{1}{D}$ etc. And in like manner $0=$ the aggregate of the products of four terms of the series and $+\frac{1}{1\cdot 2\cdot 3\cdot 4\cdot 5y}=$ the aggregate of products of five terms of this series, and so forth.

§7 But having put the smallest arc $AM=A$, whose sine is $PM=y$, and having put the half of the circumference of the circle $=p$

$$A,\quad p+A,\quad 2p-A,\quad 3p+A,\quad 4p-A,\quad 5p+A,\quad 6p+A\quad \text{etc}$$

likewise

$$-p-A,\quad -2p+A,\quad -3p-A,\quad -4p+A,\quad -5p-A\quad \text{etc.}$$

will be all the arcs, whose sine is the same, namely y. Therefore, the series we assumed before

$$\frac{1}{A},\quad \frac{1}{B},\quad \frac{1}{C},\quad \frac{1}{D}\quad \text{etc.}$$

is transformed into this one

$$\frac{1}{A},\ \frac{1}{p-A},\ \frac{1}{-p-A},\ \frac{1}{2p+A},\ \frac{1}{-2p+A},\ \frac{1}{3p-A},\ \frac{1}{-3p-A},\ \frac{1}{4p+A},\ \frac{1}{-4p+A}\ \text{etc.}$$

Therefore, the sum of all these terms is $=\frac{1}{y}$, but the sum of the products of two terms of this series is equal to 0, the sum of the products of three is $=\frac{-1}{1\cdot 2\cdot 3y}$, the sum of the products of four is $=0$, the sum of products of five is $=\frac{+1}{1\cdot 2\cdot 3\cdot 4\cdot 5y}$, the sum of products of six is $=0$. And so forth.

§8 But if one has an arbitrary series

$$a+b+c+d+e+f+\text{etc.},$$

whose sum we want to put $=\alpha$, the sum of products of two terms $=\beta$, the sum of products of three $=\gamma$, the sum of products of four $=\delta$ etc., the sum of the squares of the single terms will be

$$a^2+b^2+c^2+d^2+\text{etc.}=\alpha^2-2\beta,$$

the sum of the cubes on the other hand

$$a^3+b^3+c^3+d^4+\text{etc.}=\alpha^3-3\alpha\beta+3\gamma,$$

the sum of the forth powers

$$=\alpha^4-4\alpha^2\beta+4\alpha\gamma+2\beta^2-4\delta.$$

But that it becomes more clear, how these formulas proceed, let us assume that the sum of the terms a, b, c, d etc. is $=P$, the sum of the squares is $=Q$, the sum of the cubes is $=R$, the sum of the forth powers $=S$, the sum of the fifth powers $=T$, the sum of the sixth $=V$ etc. Having constituted all this it will be

$$P=\alpha,\quad Q=P\alpha-2\beta,\quad R=Q\alpha-P\beta+3\gamma,\quad S=R\alpha-Q\beta+P\gamma-4\delta,$$
$$T=S\alpha-R\beta+Q\gamma-P\delta+5\varepsilon\quad \text{etc.}$$

§9 Therefore, since in our case the sum of all terms of the series

$$\frac{1}{A},\quad \frac{1}{p-A},\quad \frac{1}{-p-A},\quad \frac{1}{2p+A},\quad \frac{1}{-2p+A},\quad \frac{1}{3p-A},\quad \frac{1}{-3p-A}\quad \text{etc.}$$

is α or $=\frac{1}{y}$, the sum of the products of two or $\beta=0$ and further

$$\gamma=\frac{-1}{1\cdot 2\cdot 3y},\quad \delta=0,\quad \varepsilon=\frac{+1}{1\cdot 2\cdot 3\cdot 4\cdot 5y},\quad \zeta=0\quad \text{etc.},$$

the sum of those terms itself will be

$$P=\frac{1}{y},$$

the sum of the squares of those terms will be

$$Q=\frac{P}{y}=\frac{1}{y^2},$$

the sum of the cubes of those terms will be

$$R = \frac{Q}{y} - \frac{1}{1 \cdot 2y},$$

the sum of the forth powers will be

$$S = \frac{R}{y} - \frac{P}{1 \cdot 2 \cdot 3y}$$

and further

$$\begin{aligned}
T &= \frac{S}{y} - \frac{Q}{1 \cdot 2 \cdot 3y} + \frac{1}{1 \cdot 2 \cdot 3 \cdot 4y},\\
V &= \frac{T}{y} - \frac{R}{1 \cdot 2 \cdot 3y} + \frac{P}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5y},\\
W &= \frac{V}{y} - \frac{S}{1 \cdot 2 \cdot 3y} + \frac{Q}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5y} - \frac{1}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6}.
\end{aligned}$$

Using this law the sums of the remaining higher powers are easily determined.

§10 Now let us put the sine $PM = y$ equal to the radius, that it is $y = 1$; the smallest arc A, whose sine is 1, will be the fourth part of the circumference $= \frac{1}{2}p$, or while q denotes the fourth part of the circumference it will be $A = q$ and $p = 2q$. Therefore, the upper series will go over into this one

$$\frac{1}{q}, \quad \frac{1}{q}, \quad -\frac{1}{3q}, \quad -\frac{1}{3q}, \quad +\frac{1}{5q}, \quad +\frac{1}{5q}, \quad -\frac{1}{7q}, \quad -\frac{1}{7q}, \quad +\frac{1}{9q}, \quad +\frac{1}{9q} \quad \text{etc.}$$

in which each two terms are equal. Therefore, the sum of these terms, which is

$$\frac{2}{q}\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \text{etc.}\right),$$

is equal to $P = 1$. Therefore, hence this equation results

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \text{etc.} = \frac{q}{2} = \frac{p}{4}.$$

Therefore, the quadruple of this series becomes equal to the half of the circumference of the circle, whose radius is 1, or to the circumference of the whole circle, whose diameter is 1. And this is the series found first by Leibniz some time ago and used by him to define the quadrature of the circle. From this example the general foundation of this method will become evident for everybody so that the remaining results we will derive using this method cannot be in any doubt.

§11 Now let us take the squares of the found terms for the case, in which it is $y = 1$, and this series will result

$$+\frac{1}{q^2}+\frac{1}{q^2}+\frac{1}{9q^2}+\frac{1}{9q^2}+\frac{1}{25q^2}+\frac{1}{25q^2}+\text{etc.},$$

whose sum is

$$\frac{2}{q^2}\left(\frac{1}{1}+\frac{1}{9}+\frac{1}{25}+\frac{1}{49}+\text{etc.}\right),$$

which therefore must be equal to $Q = P = 1$. Hence it follows that the sum of this series

$$1+\frac{1}{9}+\frac{1}{25}+\frac{1}{49}+\text{etc.}$$

is $= \frac{q^2}{2} = \frac{p^2}{8}$, while p denotes the whole circumference of the circle, whose diameter is $= 1$. But the sum of this series

$$1+\frac{1}{9}+\frac{1}{25}+\text{etc.}$$

depends on the sum of the series

$$1+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+\frac{1}{25}+\text{etc.},$$

since this diminished by its fourth part gives the first series. Therefore, the sum of this series is equal to the sum of that series together with is third part. Therefore, it will be

$$1+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+\frac{1}{25}+\frac{1}{36}+\text{etc.}=\frac{p^2}{6},$$

and hence the sum of this series multiplied by 6 is equal to the square of the circumference of the circle, whose diameter is 1; this is the proposition itself I mentioned at the beginning.

§12 Since therefore in the case, in which it is $y = 1$, it is $P = 1$ and $Q = 1$, the values of remaining letters R, S, T, V etc. will be as follows

$$R = \frac{1}{2}, \quad S = \frac{1}{3}, \quad T = \frac{5}{24}, \quad V = \frac{2}{15}, \quad W = \frac{61}{720}, \quad X = \frac{17}{315} \quad \text{etc.}$$

But since the sum of the cubes is equal to $R = \frac{1}{2}$, it will be

$$\frac{2}{q^3}\left(1 - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} + \frac{1}{9^3} - \text{etc.}\right) = \frac{1}{2}.$$

Hence it will be

$$1 - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} + \frac{1}{9^3} - \text{etc.} = \frac{q^3}{4} = \frac{p^3}{32}.$$

Therefore, the sum of this series multiplied by 32 gives the cube of the circumference of the circle, whose diameter is 1. In like manner the sum of the fourth powers, which is

$$\frac{2}{q^4}\left(1 + \frac{1}{3^4} + \frac{1}{5^4} + \frac{1}{7^4} + \frac{1}{9^4} + \text{etc.}\right)$$

must be equal to $\frac{1}{3}$ and hence it will be

$$1 + \frac{1}{3^4} + \frac{1}{5^4} + \frac{1}{7^4} + \frac{1}{9^4} + \text{etc.} = \frac{q^4}{6} = \frac{p^4}{96}.$$

But on the other hand this series multiplied by $\frac{16}{15}$ is equal to this one

$$1 + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \frac{1}{5^4} + \frac{1}{6^4} + \text{etc.};$$

hence this series is equal to $\frac{p^4}{90}$, or the sum of this series

$$1 + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \text{etc.}$$

multiplied by 90 gives the fourth power of the circumference of the circle, whose diameter is 1.

§13 In like manner the sum of the higher powers will be found; one will find the following equations

$$1-\frac{1}{3^5}+\frac{1}{5^5}-\frac{1}{7^5}+\frac{1}{9^5}-\text{etc.}=\frac{5q^5}{48}=\frac{5p^5}{1536}$$

and

$$1+\frac{1}{3^6}+\frac{1}{5^6}+\frac{1}{7^6}+\frac{1}{9^6}+\text{etc.}=\frac{q^6}{15}=\frac{p^6}{960}.$$

But having found the sum of this series at the same time the sum of this series will be known

$$1+\frac{1}{2^6}+\frac{1}{3^6}+\frac{1}{4^6}+\frac{1}{5^6}+\text{etc.},$$

which will be

$$=\frac{p^6}{945}.$$

Further, for the seventh powers it will be

$$1-\frac{1}{3^7}+\frac{1}{5^7}-\frac{1}{7^7}+\frac{1}{9^7}-\text{etc.}=\frac{61q^7}{1440}=\frac{61p^7}{184320}$$

and for the eighth

$$1+\frac{1}{3^8}+\frac{1}{5^8}+\frac{1}{7^8}+\frac{1}{9^8}+\text{etc.}=\frac{17q^8}{630}=\frac{17p^8}{161280},$$

whence one deduces

$$1+\frac{1}{2^8}+\frac{1}{3^8}+\frac{1}{4^8}+\frac{1}{5^8}+\frac{1}{6^8}+\text{etc.}=\frac{p^8}{9450}.$$

But it is seen that in these series in the powers of odd exponents the signs of the terms alternate, for the even powers on the other hand they are equal; and this is the reason that the sum of this general series

$$1+\frac{1}{2^n}+\frac{1}{3^n}+\frac{1}{4^n}+\text{etc.}$$

can only be exhibited in the cases, in which n is an even number. Furthermore, it is also to be noted, if the general term of the series $1, 1, \frac{1}{2}, \frac{1}{3}, \frac{5}{24}, \frac{2}{15}, \frac{61}{720}, \frac{17}{315}$ etc., which values we found for the letters P, Q, R, S etc., could be assigned, then then the quadrature of the circle would be exhibited by it.

§14 In the consideration up to this point we put the sine PM equal to the radius; therefore, let us see, what kind of series result, if other values are attributed to y. Therefore, let $y = \frac{1}{\sqrt{2}}$; the smallest arc corresponding to this sine is $\frac{1}{4}p$. Therefore, having put $A = \frac{1}{4}p$ the sum of the simple terms or the first power will be this one

$$\frac{4}{p} + \frac{4}{3p} - \frac{4}{5p} - \frac{4}{7p} + \frac{4}{9p} + \frac{4}{11p} - \text{etc.};$$

the sum P of this series is $\frac{1}{y} = \sqrt{2}$. Therefore, one will have

$$\frac{p}{2\sqrt{2}} = 1 + \frac{1}{3} - \frac{1}{5} - \frac{1}{7} + \frac{1}{9} + \frac{1}{11} - \frac{1}{13} - \frac{1}{15} + \text{etc.},$$

which series only differs from Leibniz's series with respect to the sings and was given by Newton a long time ago. But the sum of the squares of those terms, namely

$$\frac{16}{p^2}\left(1 + \frac{1}{9} + \frac{1}{25} + \frac{1}{49} + \text{etc.}\right),$$

is equal to $Q = 2$. Therefore, it will be

$$1 + \frac{1}{9} + \frac{1}{25} + \frac{1}{49} + \text{etc.} = \frac{p^2}{8},$$

as it was found before.

§15 If it is $y = \frac{\sqrt{3}}{2}$, the smallest arc corresponding to this sine will be $60°$ and hence $A = \frac{1}{3}p$. Therefore, in this case the following series of terms results

$$\frac{3}{p} + \frac{3}{2p} - \frac{3}{4p} - \frac{3}{5p} + \frac{3}{7p} + \frac{3}{8p} - \text{etc.};$$

the sum of these terms is equal to $\frac{1}{y} = \frac{2}{\sqrt{3}}$. Therefore, one will have

$$\frac{2p}{3\sqrt{3}} = 1 + \frac{1}{2} - \frac{1}{4} - \frac{1}{5} + \frac{1}{7} + \frac{1}{8} - \frac{1}{10} - \frac{1}{11} + \text{etc.}$$

But the sum of the squares of those terms is $= \frac{1}{y^2} = \frac{4}{3}$; hence it follows that it will be

$$\frac{4p^2}{27} = 1 + \frac{1}{4} + \frac{1}{16} + \frac{1}{25} + \frac{1}{49} + \frac{1}{64} + \text{etc.},$$

in which series the terms containing three are missing. But this series also depends on this one

$$1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} \quad \text{etc.},$$

whose sum had been found to be $= \frac{p^2}{6}$; for, if this series is diminished by its ninth part, the series found above results, whose sum must therefore be $= \frac{p^2}{6}\left(1 - \frac{1}{9}\right) = \frac{4pp}{27}$. In like manner, if other sines are assumed, other series will result, so of the simple terms as of the squares and the higher powers of the terms; whose sum involve the quadrature of the circle.

§16 But if one puts $y = 0$, series of this kind cannot further be assigned because of the y in the denominator or the initial equation divided by y. But series can hence be deduced in another way; since these are the series

$$1 + \frac{1}{2^n} + \frac{1}{3^n} + \frac{1}{4^n} + \text{etc.}$$

themselves, if n is an even number, I will deduce how the sum of these series are to be found separately from this case, in which it is $y = 0$. But having put $y = 0$ the fundamental equation goes over into this one

$$0 = s - \frac{s^3}{1 \cdot 2 \cdot 3} + \frac{s^5}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} - \frac{s^7}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7} + \text{etc.};$$

the roots of this equation give all arcs, whose sine is $= 0$. But one and the smallest root is $s = 0$, whence the equation divided by s will exhibit all remaining arcs, whose sine is $= 0$, which arcs will equally be the roots of this equation

$$0 = 1 - \frac{s^2}{1 \cdot 2 \cdot 3} + \frac{s^4}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} - \frac{s^6}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7} + \text{etc.}$$

But the arcs, whose sine is $= 0$, are

$$p, \quad -p, \quad +2p, \quad -2p, \quad 3p, \quad -3p \quad \text{etc.};$$

the one of each two is the negative of the other, what also the symmetry of the equation implies. Hence the divisors of that equation will be

$$1 - \frac{s}{p}, \quad 1 + \frac{s}{p}, \quad 1 - \frac{s}{2p}, \quad 1 + \frac{s}{2p} \quad \text{etc.}$$

and by combing each two of these terms it will be

$$1-\frac{s^2}{1\cdot 2\cdot 3}+\frac{s^4}{1\cdot 2\cdot 3\cdot 4\cdot 5}-\frac{s^6}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7}+\text{etc.}$$
$$=\left(1-\frac{s^2}{p^2}\right)\left(1-\frac{s^2}{4p^2}\right)\left(1-\frac{s^2}{9p^2}\right)\left(1-\frac{s^2}{16p^2}\right)\text{etc.}$$

§17 It is now manifest from the nature of equations that the coefficient of ss or $\frac{1}{1\cdot 2\cdot 3}$ will be equal to

$$\frac{1}{p^2}+\frac{1}{4p^2}+\frac{1}{9p^2}+\frac{1}{16p^2}+\text{etc.}$$

The sum of the products of two terms of this series will be $=\frac{1}{1\cdot 2\cdot 3\cdot 4\cdot 5}$ and the sum of the products of three will be $=\frac{1}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7}$ etc. Therefore, according to § 8 it will be

$$\alpha=\frac{1}{1\cdot 2\cdot 3}, \quad \beta=\frac{1}{1\cdot 2\cdot 3\cdot 4\cdot 5}, \quad \gamma=\frac{1}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7} \quad \text{etc.}$$

and having put the sum of the terms

$$\frac{1}{p^2}+\frac{1}{4p^2}+\frac{1}{9p^2}+\frac{1}{16p^2}+\text{etc.}=P$$

and the sum of the squares of the same terms $=Q$, the sum of the cubes $=R$, the sum of the forth powers $=S$ etc., it will be by means of § 8

$$\begin{aligned}
P &= \alpha = \frac{1}{1\cdot 2\cdot 3} = \frac{1}{6},\\
Q &= P\alpha - 2\beta = \frac{1}{90},\\
R &= Q\alpha - P\beta + 3\gamma = \frac{1}{945},\\
S &= R\alpha - Q\beta + P\gamma - 4\delta = \frac{1}{9450},\\
T &= S\alpha - r\beta + Q\gamma - P\delta + 5\varepsilon = \frac{1}{93555}\\
V &= T\alpha - S\beta + R\gamma - Q\gamma + P\varepsilon - 6\zeta = \frac{691}{6825\cdot 93555}\\
&\text{etc.}
\end{aligned}$$

§18 Therefore, using these relations the following sums are derived

$$\begin{aligned}
1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\frac{1}{5^2}+\text{etc.} &= \frac{p^2}{6} = P',\\
1+\frac{1}{2^4}+\frac{1}{3^4}+\frac{1}{4^4}+\frac{1}{5^4}+\text{etc.} &= \frac{p^4}{90} = Q',\\
1+\frac{1}{2^6}+\frac{1}{3^6}+\frac{1}{4^6}+\frac{1}{5^6}+\text{etc.} &= \frac{p^6}{945} = R',\\
1+\frac{1}{2^8}+\frac{1}{3^8}+\frac{1}{4^8}+\frac{1}{5^8}+\text{etc.} &= \frac{p^8}{9450} = S',\\
1+\frac{1}{2^{10}}+\frac{1}{3^{10}}+\frac{1}{4^{10}}+\frac{1}{5^{10}}+\text{etc.} &= \frac{p^{10}}{93555} = T',\\
1+\frac{1}{2^{12}}+\frac{1}{3^{12}}+\frac{1}{4^{12}}+\frac{1}{5^{12}}+\text{etc.} &= \frac{691p^{12}}{6825\cdot 93555} = V';\\
&\text{etc.,}
\end{aligned}$$

these series can be continued to the higher powers with a lot of work using the given rule. But by dividing the single series by the preceding ones the following equations will result

$$p^2 = 6P' = \frac{15Q'}{P'} = \frac{21R'}{2Q'} = \frac{10S'}{R'} = \frac{99T'}{10S'} = \frac{6825V'}{691T'} \quad \text{etc.;}$$

and the square of the circumference of the circle, whose diameter is 1, becomes equal to all these single expressions.

§19 But since the sum of these series, even though they can easily be exhibited approximately, nevertheless are quite useless to express the circumference of this circle approximately because of the square root, which would have to be extracted, from the first series we will find expressions, which are equal to the circumference p itself. For, we will find the following expressions for p:

$$
\begin{aligned}
p = {} & 4\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \text{etc.}\right) \\
p = {} & 2 \cdot \frac{1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{9^2} + \frac{1}{11^2} + \text{etc.}}{1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \text{etc.}} \\
p = {} & 4 \cdot \frac{1 - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} + \frac{1}{9^3} - \frac{1}{11^3} + \text{etc.}}{1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{9^2} + \frac{1}{11^2} + \text{etc.}} \\
p = {} & 3 \cdot \frac{1 + \frac{1}{3^4} + \frac{1}{5^4} + \frac{1}{7^4} + \frac{1}{9^4} + \frac{1}{11^4} + \text{etc.}}{1 - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} + \frac{1}{9^3} - \frac{1}{11^3} + \text{etc.}} \\
p = {} & \frac{16}{5} \cdot \frac{1 - \frac{1}{3^5} + \frac{1}{5^5} - \frac{1}{7^5} + \frac{1}{9^5} - \frac{1}{11^5} + \text{etc.}}{1 + \frac{1}{3^4} + \frac{1}{5^4} + \frac{1}{7^4} + \frac{1}{9^4} + \frac{1}{11^4} + \text{etc.}} \\
p = {} & \frac{25}{8} \cdot \frac{1 + \frac{1}{3^6} + \frac{1}{5^6} + \frac{1}{7^6} + \frac{1}{9^6} + \frac{1}{11^6} + \text{etc.}}{1 - \frac{1}{3^5} + \frac{1}{5^5} - \frac{1}{7^5} + \frac{1}{9^5} - \frac{1}{11^5} + \text{etc.}} \\
p = {} & \frac{192}{61} \cdot \frac{1 - \frac{1}{3^7} + \frac{1}{5^7} - \frac{1}{7^7} + \frac{1}{9^7} - \frac{1}{11^7} + \text{etc.}}{1 + \frac{1}{3^6} + \frac{1}{5^6} + \frac{1}{7^6} + \frac{1}{9^6} + \frac{1}{11^6} + \text{etc.}}
\end{aligned}
$$